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ON GROUP ALGEBRAS OF FINITE GROUPS

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In this note we study the group algebra KG of a finite p-solvable group G over a field K of characteristic $p > 0$. Let $J(KG)$ be the Jacobson radical of KG, and let $t(G)$ be the least positive integer t such that $J(KG)^t = 0$. Since $J(EG) = E \otimes_K J(KG)$ for any extension field E of K, we may assume that K is algebraically closed. We would like to know the relation between $t(G)$ and the structure of G. When $t(G) \leqslant 3$, p-solvable groups G are completely determined by D.A.R. Wallace ([9], [10]) and K. Motose and Y. Ninomiya [7]. The purpose of this note is to determine the structure of p-solvable groups G with $t(G) = 4$ under the assumption that $O_{p'}(G)$ are abelian.

We shall use the following notation. For a positive integer n let S_n and A_n be the symmetric group and the alternating group of degree n, respectively. Let $O_{p'}(G)$ and $O^{p'}(G)$ be the maximal normal subgroup of G of order prime to p and the minimal normal subgroup of G of index prime to p, respectively. Following custom we write $O(G)$ and $O'(G)$ for $O_{2'}(G)$ and $O^{2'}(G)$, respectively. For a ring R and a positive integer n let $(R)_n$ be the ring of all $n \times n$ matrices with entries in R. We use the other notation following Gorenstein's book [3].

By making use of [9, Theorem], [2, Theorem 1] and [10, Theorem 3.3] we have

Proposition 1. If G is a finite p-solvable group with a p-Sylow subgroup P and if $t(G) = 4$, then $p = 2$ and one of the following holds;

(i) P is cyclic of order 4,

(ii) P is elementary abelian of order 8,

(iii) $G/O(G) \cong S_4$.

Remark 1. The converse of Proposition 1 does not hold in general (see Motose's example [6, Example 2]). However, the following holds.

Proposition 2. If $p = 2$ and if G is a finite 2-solvable group with a 2-Sylow subgroup P which satisfies one of the following;

(i) P is cyclic of order 4,

(ii) P is elementary abelian of order 8,

(iii) $G = S_4$,

then $t(G) = 4$.

Because of Propositions 1 and 2 we assume in the rest of this note that

$$p = 2 \quad \text{and} \quad G/O(G) \cong S_4.$$

Then a 2-Sylow subgroup P of G is dihedral of order 8. Thus, by [3, Theorem 7.7.3], P has subgroups X and Y such that X and Y are both noncyclic of order 4, $X \underset{P}{\not\sim} Y$, $|N_G(X):C_G(X)| = 6$ and $|N_G(Y):C_G(Y)| = 2$.

By [4, V 25.12 Satz, V 25.7 Satz und V 25.3 Satz], [8, Lemma 2.1] and [11, Proposition 3.2], we have

Lemma 1. If U is a subgroup of S_4 and if K^cU is a twisted group algebra of U over K with respect to the factor set c, then $K^cU \cong KU$ as K-algebras.

By making use of Lemma 1, [5, Theorem 2] and [1] we obtain the following two lemmas.

Lemma 2. $t(G) = 4$ if and only if $t(N_G(X)) = 4$.

Lemma 3. If $X \triangleleft G$, then

$$KG \cong \left(\bigoplus_{i=1}^{m}(KS_4)_{\alpha_i}\right) \oplus \left(\bigoplus_{j=1}^{n/2}(KA_4)_{\beta_j}\right) \oplus \left(\bigoplus_{k=1}^{u/3}(KP)_{\gamma_k}\right) \oplus \left(\bigoplus_{\ell=1}^{v/6}(KX)_{\delta_\ell}\right)$$

as K-algebras for positive integers α_i, β_j, γ_k and δ_ℓ where m, n, u and v are the numbers of irreducible complex characters ψ of $O(G)$ such that $I_G(\psi)/O(G) \cong S_4$, A_4, P and X, respectively, and $I_G(\psi)$ is the inertia group of ψ in G.

From the above lemmas we have the following main result.

Theorem. Let $M = O'(N_G(X))$. If $O(M)$ is abelian, then the following are equivalent:

(1) $t(G) = 4$.

(2) $t(M) = 4$.

(3) $|C_M(P)| = 2$ where P is a 2-Sylow subgroup of M.

(4) When $g \in M$ such that $|gO(M)| = 3$ in $M/O(M)$, we have $g \in C_M(O(M))$.

Remark 2. In Theorem for the case where $O(M)$ is nonabelian (2) and (3) are not equivalent in general.

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